Grader's Note

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1 Comments

1.1 Don't use limit naively

One usual mistake I see when I grade papers is naive and improper use of limit. Let's see uncommon but notable mistake. Let $\{O_n\}_n$ be the family of open intervals in $\mathbb{R}$ such that $O_n = (-\frac{1}{n}, \frac{1}{n})$. Does there exist any concept of limit on this family? Indeed does, we have $\cap_{i=1}^{\infty} O_i$, because $\{O_n\}_n$ is decreasing sequence of intervals. Some of you think $\cap_{i=1}^{\infty} O_i$ as like as $\lim_{n\to\infty} O_n$. (Note we don't use such a expression) Since

$$\cap_{i=1}^{\infty} O_i = \{x \in \mathbb{R} | x \in O_i \text{ for all } i \in \mathbb{N}\}, \tag{1}$$

we have $\cap_{i=1}^{\infty} O_i = \{0\}$. However, one student had this approach:

$$\begin{aligned} \cap_{i=1}^{\infty} O_i &= \lim_{n\to\infty} O_n & (2)\\ &= \lim_{n\to\infty} (-\frac{1}{n}, \frac{1}{n}) & (3)\\ &= (\lim_{n\to\infty} -\frac{1}{n}, \lim_{n\to\infty} \frac{1}{n}) & (4)\\ &= (0,0) = \emptyset & (5) \end{aligned}$$

There are several problems in this approach. First, understanding $\cap_{i=1}^{\infty} O_i$ as a limit (2) is problematic. As you can see in (1), there is no limit involved, in definition. Actually, infinity in (1) is just a notation: $\cap_{i=1}^{\infty} O_i$ is just a $\cap_{i\in\mathbb{N}} O_i$, where $\mathbb{N}$ is just a set of natural numbers. $\mathbb{N}$ is an index set of $\{O_n\}_n$.

From (3) to (4), this student put limit in parenthesis. Since result is wrong, this approach is anyway wrong, but you should understand this: *Never exchange the order of limit and other operations if you are not sure it is possible.* Changing the order of operation has never been trivial. Many parts of studying analysis consist of to study when you can change the order of limit and other operations: If you study Lebesgue integral theory later, you'll be startled to see the biggest result of Lebesgue integral theory consists of the exchange the order of limit and integral. In this course, the story is not that different. Why are you studying 'uniform convergence' of functions? It's because pointwise convergence is not enough to deal with limit.

I'll put it here again: *Never exchange the order of limit and other operations if you are not sure it is possible.*

1.2 Compactness: is this closed under countable union?

There was a homework problem using the fact that continuous function on a compact set is uniformly continuous on that compact set. One interesting approach was this:

1. $f(x)$ is continuous on $\mathbb{R}$.
2. $f(x)$ is continuous on a compact subset $C_n = [-n, n]$ of $\mathbb{R}$, for any $n \in \mathbb{N}$.
3. Thus $f(x)$ is uniformly continuous on C_n for any $n \in \mathbb{N}$.
4. Therefore, $f(x)$ is uniformly continuous on $\mathbb{R} = \cup_{n\in\mathbb{N}} C_n$.

This seems a bit natural; especially if you are not much accustomed to mathematics. My second advice is here: *When it goes infinity, take care what's happening.* Unfortunatelly, the approach above is not valid; let me give you a counterexample of this approach.

0. Let $f(x) = \frac{1}{x}$ on $(0, \infty)$.
1. $f(x)$ is continuous on $(0, \infty)$.
2. $f(x)$ is continuous on a compact subset $C_n = [\frac{1}{n}, n]$ of $(0, \infty)$, for any $n \in \mathbb{N}$.
3. Thus $f(x)$ is uniformly continuous on C_n for any $n \in \mathbb{N}$.
4. However, $f(x)$ is not uniformly continuous on $(0, \infty) = \cup_{n\in\mathbb{N}} C_n$.

So now you understand this approach doesn't hold. Why do you think this is false? You may think about this. Most at all, was closed set closed under countable union? On the other hand, do you think 'boundedness' is related to this fallacy? How about $1/(1 - x^2)$ on $(-1, 1)$?